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On inviscid flow in a waterfall

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A nonlinear steady state solution is obtained for an incompressible, inviscid fluid in a waterfall under the action of gravity, using a special case of the system of differential equations of a recent theory of fluid sheets derived by a direct approach. This analytical solution has a simple form and offers some advantages over previous solutions of the problem either by asymptotic techniques or by numerical procedures. A comparison with experimental results is also indicated.

1. Introduction

This paper is concerned with two-dimensional motion of an incompressible, inviscid fluid in a waterfall under the action of gravity. A nonlinear steady state solution of the problem is obtained using a special case of the system of differential equations of a *Cosserat* (or a *directed*) fluid sheet contained in two recent papers by Green & Naghdi (1976*a*, 1977). Before proceeding further, it is desirable for later reference to provide here the following statement of the problem.

Statement of the problem. Consider the steady two-dimensional flow of an incompressible, inviscid fluid under the action of gravity over a bed (or a cliff) leading to a free overfall (see figure 1); the effect of surface tension is assumed negligible. Two distinct regions of flow on either side of the edge of the fluid bed may be associated with this problem: The upstream region (labelled as I in figure 1) is characterized by a free top surface and a fixed level bottom, while in the downstream region (labelled as II in figure 1) both the top and bottom surfaces of the fluid are free. Far upstream of the bed's edge the fluid is assumed to flow as a uniform stream, while downstream the fluid falls freely under the action of gravity. Of particular interest in analysing the problem is the prediction of the height of the free surface in the upstream region and the determination of the downstream solution, i.e. the shape of the free surfaces including the vertical thickness, as well as the velocity and hence the flow rate.

It should be clear from the above *statement* that the rather complex nature of the flow in both upstream and downstream regions is due to two major factors: (1) The location of the free surfaces are unknown and are determined by the solution of the problem and (2) the speed of a fluid particle on any of the free surfaces is not constant as a result of gravitational acceleration. Although the overfall problem is one of the simplest problems which includes a double free surface region (downstream of the fluid bed), even in this case an exact analytical solution of the problem has not been possible so far with the use of the three-dimensional equations of an incompressible, inviscid fluid. For this reason, previous authors have discussed the solution of the

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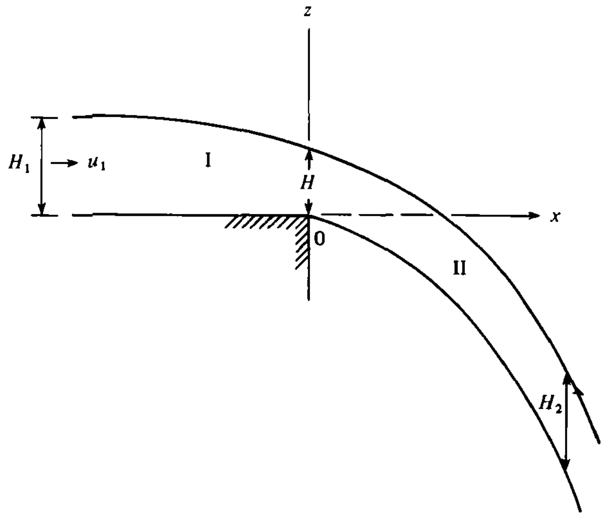

FIGURE 1. A sketch of the free overfall in the x–z plane of a rectangular Cartesian co-ordinate system showing the upstream and downstream regions labelled as I and II, respectively. Also shown are the vertical height H_1 and horizontal velocity u_1 far upstream, the vertical thickness H_2 far downstream and the height H at the edge $x = 0$ of the fluid bed.

problem either by numerical procedures or by asymptotic techniques. Keller & Weitz (1957) have developed a series in powers of the jet thickness and obtained a solution for the region downstream of the fluid bed only. Clarke (1965) has solved the problem of overfall for large Froude numbers with the use of matched asymptotic techniques and by utilizing an asymptotic expansion based on the reciprocal of Froude number in the upstream region and another expansion based on the thinness of the fall in the downstream region. A further discussion of the problem of overfall is included in a paper of Keller & Geer (1973), who consider asymptotic solutions of a class of problems based on the slenderness ratio of the stream. An early numerical solution of the problem, employing relaxation method, was given by Southwell & Vaisey (1946); and further results, again by relaxation method, were obtained by Markland (1965). A more recent contribution to the subject is made by Chow & Han (1979), who obtain a numerical solution of the problem via the three-dimensional equations of an incompressible, inviscid fluid along with hodograph transformations.

The analytical solution of the overfall problem obtained here is quite simple. It may be contrasted with the fairly intricate numerical work of Chow & Han (1979) or the asymptotic solution of Clarke (1965) which is valid only for large Froude numbers. A significant feature of the present solution is the proof that the upstream flow *cannot* be subcritical, which confirms a speculative remark made by Chow & Han (1979, p. 3) to the effect that 'the existence of subcritical case is doubtful in steady inviscid flow'.

In order to indicate why the theory of a directed fluid sheet employed here is applicable to the free overfall problem, we first make some observations concerning the kinematical nature of this (three-dimensional) boundary-value problem. The boundary conditions require that far upstream the horizontal velocity is constant and the vertical velocity is zero, while far downstream the horizontal velocity is independent of the vertical coordinate and the pressure is constant throughout the vertical thickness. It then follows that both far upstream and far downstream, the horizontal and vertical velocities are independent of the vertical coordinate. This observation suggests that the variation of the velocity field throughout the vertical thickness of the fluid sheet is probably *small* in both regions I and II of figure 1; and is also consistent with the kinematical assumption associated with the particular two-dimensional model upon which the theory of a directed fluid sheet with a single director is constructed (Green & Naghdi 1976*a*, 1977). Indeed, as has been remarked by Green & Naghdi (1976*b*, §1, last paragraph), this kinematical assumption is equivalent to the (three-dimensional) approximation that the vertical component of the velocity field is linear in the vertical co-ordinate z and that the horizontal component of the velocity field is independent of z. In addition, it should be noted that the system of differential equations of the theory employed accounts for the effect of vertical inertia, is translation invariant and satisfies exactly the boundary conditions on the top and bottom surfaces of the fluid sheet.

2. Basic equation

We record here a special case of the nonlinear differential equations of the restricted theory of a directed fluid sheet derived by Green & Naghdi (1977) for propagation of gravity waves over a bottom of variable initial depth. These equations were derived by a direct approach based on a two-dimensional continuum model called a *Cosserat* (or a *directed*) surface. However, within the scope of the three-dimensional theory of inviscid fluids, Green & Naghdi (1976*b*) have shown that this nonlinear two-dimensional theory can also be derived from an energy equation, the incompressibility condition, invariance requirements under superposed rigid body motions, together with a single approximation for the (three-dimensional) velocity field. For additional background material on the development of this theory, reference may be made to a recent review paper by Naghdi (1979).

Initially we follow the derivation and notation of equations (2.1)–(2.5) of Naghdi & Rubin (1981; see pages 349–350 of this volume), from which, in the absence of surface tension, the condition of incompressibility and the relevant equations of motion are given by the following nonlinear partial differential equations (Green & Naghdi 1977):

$$w+\phi u_x = 0, \tag{2.6a}$$

$$\rho^*\phi\dot{u} = \hat{p}\beta_x-\bar{p}\alpha_x-p_x,\quad \rho^*\phi\dot{\lambda} = -\rho^*g\phi+\bar{p}-\hat{p},\quad \tfrac{1}{12}\rho^*\phi\dot{w} = -\tfrac{1}{2}(\hat{p}+\bar{p})+\frac{p}{\phi}, \tag{2.6b, c, d}$$

where the subscripts denote partial differentiation with respect to x, and α = vertical location of the bottom fluid surface relative to a fixed system of Cartesian co-ordinate axes (x, y, z). Also, the position vectors at the bottom and top surfaces of the fluid sheet may be expressed as

$$\bar{\mathbf{p}} = x\mathbf{e}_1+\alpha(x,t)\,\mathbf{e}_3,\quad \hat{\mathbf{p}} = x\mathbf{e}_1+\beta(x,t)\,\mathbf{e}_3. \tag{2.7a, b}$$

For steady state motions, the equations of motion (2.6) reduce to

$$(\phi u)_x = 0, \tag{2.8a}$$

$$\rho^* \phi u u_x = \hat{p}\beta_x - \overline{p}\alpha_x - p_x, \quad \rho^*\phi u \lambda_x = -\rho^* g\phi + \overline{p} - \hat{p}, \quad \tfrac{1}{12}\rho^*\phi u w_x = -\tfrac{1}{2}(\hat{p}+\overline{p}) + \frac{p}{\phi}. \tag{2.8b, c, d}$$

As noted above, Green & Naghdi (1976*b*) have shown that the theory which results in the system of equations (2.6) can also be derived from the three-dimensional theory by approximating the position vector in the form

$$\mathbf{p}^* = \mathbf{r} + \theta^3 \phi \mathbf{e}_3, \tag{2.9}$$

where θ^3 is a convected co-ordinate. Corresponding to the values $\theta^3 = \pm\frac{1}{2}$ the expression (2.9) locates the top and bottom surfaces, respectively. With the help of (2.7), (2.8) and (2.9), it can then be shown that ψ and ϕ are related to β and α by

$$\psi = \tfrac{1}{2}(\beta+\alpha), \quad \phi = \beta - \alpha. \tag{2.10a, b}$$

For later reference, we also recall an expression for p in terms of the pressure p^* in the three-dimensional theory (see (4.20) in Green & Naghdi 1976*b*), namely

$$p = \phi \int_{-\frac{1}{2}}^{\frac{1}{2}} p^* d\theta^3, \tag{2.11}$$

where the limits of integration correspond to the bottom and top surfaces of the fluid sheet indicated above.

As noted previously (see, e.g. Naghdi 1979), a direct theory of the type utilized in this paper can only provide partial information in some sense: for example, in the case of fluid sheets, information concerning quantities which can be regarded as representing the medium response confined to a surface or its neighbourhood as a consequence of the (three-dimensional) motion of the fluid. Moreover, it should be noted that the theory of a directed fluid sheet as its basic kinetical ingredients admits forces and couples (rather than stresses), which can be interpreted as resultants, and is employed here mainly for the purpose of obtaining satisfactory predictions for the shape of the free surfaces and such quantities as the amplitude of the motion and flow rate. It should not be expected to provide accurate information about stress distributions (here the pressure) across the thickness of the fluid sheet as is also evident from (2.11).

3. Formulation of the problem

A statement of the problem under consideration is given in section 1. With reference to figure 1, we take the origin of the x–z co-ordinate axes to coincide with the corner of the edge of the fluid bed and divide the region into two parts, namely (i) the upstream region $x \leqq 0$ (labelled as I in figure 1) and (ii) the downstream region $x > 0$ (labelled as II in figure 1). It follows that (in the absence of surface tension) the pressure $\hat{p}$ at the top surface equals the atmospheric pressure p_0 in both the upstream and the downstream regions, while the pressure $\overline{p}$ at the bottom surface equals p_0 only in the downstream region and is to be determined in the upstream region. Also, the bottom of the fluid bed is level in the upstream region, while in the downstream region the vertical location of the bottom surface α in (2.7) is unknown and must be determined in the course of solution. These preliminary observations may be summarized as†:

$$\text{Region I } (x \leqq 0) \quad \left.\begin{cases} \hat{p} = p_0, & \bar{p} \text{ to be determined,} \\ \alpha = 0, & \end{cases}\right\} \tag{3.1}$$

and

$$\text{Region II } (x > 0) \quad \left.\begin{cases} \hat{p} = p_0, & \bar{p} = p_0, \\ \alpha \text{ to be determined.} & \end{cases}\right\} \tag{3.2}$$

With the help of (2.10), (3.1) and (3.2), the steady state equations of motion (2.8) for the two regions I and II can be reduced to the forms:

Region I:

$$\phi u = k_1, \quad \rho^* k_1^2 \left(\frac{1}{\phi}\right)_x = -P_x, \quad \tfrac{1}{2}\rho^* k_1^2 \left(\frac{\phi_x}{\phi}\right)_x = -\rho^* g\phi + \bar{P}, \quad \tfrac{1}{12}\rho^* k_1^2 \left(\frac{\phi_x}{\phi}\right)_x = \frac{P}{\phi} - \tfrac{1}{2}\bar{P}. \tag{3.3a, b, c, d}$$

Region II:

$$\phi u = k_2, \quad \rho^* k_2^2 \left(\frac{1}{\phi}\right)_x = -P_x, \quad \rho^* k_2^2 \left(\frac{\psi_x}{\phi}\right)_x = -\rho^* g\phi, \quad \tfrac{1}{12}\rho^* k_2^2 \left(\frac{\phi_x}{\phi}\right)_x = \frac{P}{\phi}. \tag{3.4a, b, c, d}$$

In the above we have integrated (2.8a) to obtain (3.3a) and (3.4a) with k_1 and k_2 as constants of integration and have also introduced the notations

$$P = p - p_0\phi, \quad \bar{P} = \bar{p} - p_0. \tag{3.5a, b}$$

In order to obtain a solution which holds throughout $x(-\infty < x < \infty)$, the solutions in regions I and II must be matched at $x = 0$. This matching is accomplished by using the standard jump conditions associated with the integral balance laws of the theory of a directed fluid sheet. Assuming that the fluid sheet leaves the edge of the bed at $x = 0$ smoothly, the appropriate two-dimensional form of the jump conditions for a fluid sheet of variable initial depth may be written as‡:

$$[\![\phi u]\!] = 0, \quad [\![\rho^*\phi u^2 + p]\!] = 0, \tag{3.6a, b}$$

$$[\![\rho^*\phi u\lambda]\!] = 0, \quad [\![\tfrac{1}{12}\rho^*\phi u w]\!] = 0, \tag{3.6c, d}$$

where the notation $[\![f]\!]$ stands for

$$[\![f]\!] = f^+ - f^-, \tag{3.7}$$

with $f^+ = \lim_{x\to 0^+} f$ and $f^- = \lim_{x \to 0^-} f$. Supplementary to the jump conditions

† Since the bottom surface in region I is level, α is constant and without loss in generality we have set $\alpha = 0$ in (3.1).

‡ These jump conditions are equivalent to those used by Green & Naghdi (1976a) in a different context.

(3.6), we require both the director **d** and the vertical location of the bottom surface to be continuous across the edge of the fluid bed so that by (2.3) and (2.7) we have

$$[\![\phi]\!] = 0, \quad [\![\alpha]\!] = 0. \tag{3.8a,b}$$

After matching the solutions in regions I and II with the use of the conditions (3.6) and (3.8), we need to impose the boundary conditions associated with the flow far upstream and far downstream of the fluid bed's edge ($x = 0$). In this connexion, we suppose that far downstream the pressure distribution (in the three-dimensional theory) is uniform throughout the thickness of the fluid sheet and is equal to the atmospheric pressure p_0. Consistent with this assumption and since surface tension has been neglected, it is clear that the fluid sheet falls freely under the action of gravity. Hence, far downstream of the bed, we assume that†

$$\phi \to H_2, \quad \phi_x \to 0, \quad P \to 0 \quad \text{as} \quad x \to +\infty, \tag{3.9a,b,c}$$

where the constant vertical thickness H_2 of the fluid sheet far downstream of the bed is to be determined in the course of solution. Also, the assumption that far upstream the fluid flows as a uniform stream leads to the following boundary conditions:

$$\phi \to H_1, \quad \phi_x \to 0, \quad P = \tfrac{1}{2}\rho^* g H_1^2, \quad u \to u_1 \quad \text{as} \quad x \to -\infty, \tag{3.10a,b,c,d}$$

where the constants H_1 and u_1 denote the depth and velocity far upstream.

4. Solution

It is convenient to determine first the solution in region II. From integration of (3.4b) follows the expression

$$P = \rho^* \left(S_2 - \frac{k_2^2}{\phi}\right), \tag{4.1}$$

where S_2 is a constant of integration. Then, after substituting (4.1) into (3.4d), multiplying by ϕ_x/ϕ (since the vertical thickness ϕ is never zero), integrating and multiplying the resulting expression by $2\phi^2$, we obtain

$$\tfrac{1}{12} k_2^2 \phi_x^2 = 2R_2\phi^2 - 2S_2\phi + k_2^2, \tag{4.2}$$

where R_2 is another constant of integration. From (4.1), (4.2) and the conditions (3.9) the constants S_2 and R_2 are found to be

$$S_2 = \frac{k_2^2}{H_2}, \quad R_2 = \frac{k_2^2}{2H_2^2} \tag{4.3a,b}$$

and then (4.2) yields the following differential equation for the vertical thickness of the fluid sheet:

$$\frac{H_2^2}{12}\phi_x^2 = (\phi - H_2)^2. \tag{4.4}$$

† The condition (3.9c) is motivated from (3.5a) and (2.11) and the assumption that far downstream the (three-dimensional) pressure $p^* = p_0$.

Integrating (4.4) and using the condition (3.9a), we arrive at

$$\phi = H_2 + Ae^{-Bx}, \quad B = \frac{2\sqrt{3}}{H_2}, \tag{4.5a,b}$$

where A is a constant of integration to be determined. Next, substitution of (4.5a) into (3.4c) followed by two successive integrations yields

$$\psi = -\frac{gH_2^2}{2k_2^2}x^2 + CH_2x + D + \frac{A}{B}\frac{gH_2}{k_2^2}xe^{-Bx} - \frac{AC}{B}e^{-Bx} - \frac{A^2}{B^2}\frac{g}{2k_2^2}e^{-2Bx}, \tag{4.6}$$

where C and D are constants of integration. The solution in the downstream region given by (3.4a), (4.1), (4.5a) and (4.6) is completely determined once the constants H_2, k_2, A, C and D are evaluated.

We now turn our attention to the solution in region I. Integration of (3.3b) gives

$$P = \rho^* \left(S_1 - \frac{k_1^2}{\phi}\right), \tag{4.7}$$

where S_1 is a constant of integration. After eliminating P and $\overline{P}$ from (3.3d) with the help of (4.7) and (3.3c), multiplying by ϕ_x/ϕ, integrating and multiplying the resulting expression by $2\phi^2$, we obtain

$$\tfrac{1}{3}k_1^2\phi_x^2 = -g\phi^3 + 2R_1\phi^2 - 2S_1\phi + k_1^2, \tag{4.8}$$

where R_1 is another constant of integration. The constants k_1, S_1 and R_1, determined from (3.3a), (4.7), (4.8) and the boundary conditions (3.10), are given by

$$k_1 = H_1u_1, \quad S_1 = \tfrac{1}{2}gH_1^2 + \frac{k_1^2}{H_1}, \quad R_1 = gH_1 + \tfrac{1}{2}\frac{k_1^2}{H_1^2} \tag{4.9a,b,c}$$

and then the differential equation (4.8) can be reduced to

$$\tfrac{1}{3}k_1^2\phi_x^2 = \left(\frac{k_1^2}{H_1^2} - g\phi\right)(\phi - H_1)^2. \tag{4.10}$$

The system of equations (3.3a), (4.7), (3.3c) and (4.10) characterize the motion of fluid in the upstream region.

Before recording the solution of (4.10), it is convenient to exploit the implications of the matching conditions (3.6) and (3.8). Thus, with the help of (2.10a, b), (3.1), (3.3a), (3.4a), (4.1) and (4.7), the conditions (3.6) and (3.8) are seen to be equivalent to

$$\phi^+ = \phi^- = H, \quad \phi_x^+ = \phi_x^- = K, \quad \alpha^+ = \alpha^- = 0, \quad \alpha_x^+ = \alpha_x^- = 0 \tag{4.11a,b,c,d}$$

and

$$k_2 = k_1 = k, \quad S_2 = S_1, \tag{4.12a,b}$$

where k represents the constant rate of flow, H and K are the height and slope of the top surface of the fluid sheet at the edge $x = 0$ of the bed and where we have made use of the notations in (3.7). The constant H_2 in (3.9a) can now be determined by using (4.3a), (4.9b) and (4.12a,b) and is given by

$$H_2 = k^2\left[\tfrac{1}{2}gH_1^2+\frac{k^2}{H_1}\right]^{-1}. \tag{4.13}$$

In view of (4.12a,b), from (4.3a,b) follows the relation

$$R_2 = S_1^2/2k^2. \tag{4.14}$$

Now after evaluating (4.2) and (4.10) at $x = 0$, with the use of (4.11)–(4.13) along with elimination of the constant K in (4.11b), we obtain the cubic equation

$$gH^3+(4S_1^2/k^2-2R_1)H^2-6S_1H+3k^2 = 0 \tag{4.15}$$

for the determination of the constant vertical height H. Also, the constants A, C, D in (4.6) can be determined by using (2.11), (4.5a), (4.6) and the conditions (4.11a,b,c). The results of this calculation are:

$$A = H-H_2, \quad C = \frac{K}{2H}-\frac{A}{B}\frac{g}{k^2}, \quad D = \tfrac{1}{2}H+\frac{A}{B}\frac{K}{2H}-\frac{A^2}{B^2}\frac{g}{2k^2}. \tag{4.16a,b,c}$$

Before proceeding further, we observe that physical considerations require P to be non-negative. To see this, we first note that the pressure p^* in the three-dimensional theory of an inviscid fluid sheet is not less than the atmospheric pressure p_0, i.e. $p^* \geqq p_0$, and recall the expression for p in terms of p^* given by (2.11). It then follows from (3.5a) that

$$P = p-p_0\phi \geqq 0. \tag{4.17}$$

Moreover, with the help of (4.1) and (4.3a), this inequality implies that

$$\phi \geqq H_2. \tag{4.18}$$

In particular, when $\phi = H$ (at $x = 0$), we have $H \geqq H_2$ and hence A in (4.16a) is non-negative.

At this point, it is convenient to introduce the upstream Froude number F by

$$F^2 = \frac{k^2}{gH_1^3} \tag{4.19}$$

and define

$$\bar{\phi} = \frac{\phi}{H_1}, \quad \bar{x} = \frac{x}{H_1}, \quad \bar{H} = \frac{H}{H_1}, \quad \bar{H}_2 = \frac{H_2}{H_1}. \tag{4.20a,b,c,d}$$

In terms of the above dimensionless quantities and the Froude number F, the differential equation (4.10) and the boundary condition (3.10a) may be written as

$$\tfrac{1}{3}F^2\bar{\phi}_{\bar{x}}^2 = (F^2-\bar{\phi})(\bar{\phi}-1)^2, \quad \bar{\phi}\to 1 \quad \text{as} \quad \bar{x}\to -\infty. \tag{4.21a,b}$$

Observing that the left-hand side of (4.21a) and hence also its right-hand side is always non-negative, we conclude that

$$\{F^2-\bar{\phi}(\bar{x})\}\{\bar{\phi}(\bar{x})-1\}^2 \geqq 0. \tag{4.22}$$

Furthermore, if $\bar{\phi}(\bar{x}) \neq 1$ at a point $\bar{x}$, then it follows from (4.22) that

$$F^2 - \bar{\phi}(\bar{x}) \geqq 0 \quad (\bar{\phi}(\bar{x}) \neq 1). \tag{4.23}$$

Using (4.23) and the continuity of the function $\bar{\phi}$,it can be shown that when $F < 1$ the only solution of (4.21*a*) consistent with the boundary condition (4.21*b*) is the uniform solution $\bar{\phi}(\bar{x}) = 1$ for all $-\infty < \bar{x} \leqq 0$. On the other hand, if $F \geqq 1$, then (4.23*a*) must hold. These results may be conveniently summarized as

$$\bar{\phi}(\bar{x}) = 1 \quad \text{for} \quad F < 1, \tag{4.24a}$$

$$\bar{\phi}(\bar{x}) \leqq F^2 \quad \text{for} \quad F \geqq 1. \tag{4.24b}$$

The conclusion (4.24*a*) is, of course, a known result in cnoidal wave theory.

We are now in a position to establish the following result (by contradiction): For the boundary-value problem under consideration, the upstream Froude number F cannot be subcritical ($F < 1$), and we then conclude that the upstream flow must be either *critical* ($F = 1$) or *supercritical* ($F > 1$). By calculating the slopes ϕ_x in both regions I and II for $F < 1$ with the help of (4.24*a*), (4.5*a, b*), (4.16*a*), (4.20) and making use of the condition (4.11*b*), we can deduce the result that

$$\bar{H}_2 = \bar{H} = 1 \quad \text{for} \quad F < 1. \tag{4.25}$$

On the other hand, substitution of (4.19) and (4.20) into (4.13) yields

$$\bar{H}_2 = \frac{2F^2}{2F^2+1} < 1, \tag{4.26}$$

which is a contradiction. We have thus proved that for the boundary-value problem under consideration a solution exists only if

$$F \geqq 1. \tag{4.27}$$

Consider next the left-hand side of (4.15) and, with the help of (4.9*b, c*), (4.19) and (4.20), rewrite this as a polynomial in the form

$$Q(\bar{H}) = F^2\bar{H}^3 + (3F^4 + 2F^2 + 1)\bar{H}^2 - 3(F^2 + 2F^4)\bar{H} + 3F^4. \tag{4.28}$$

Clearly, by (4.15) the height $\bar{H}$ is determined from the equation

$$Q(\bar{H}) = 0. \tag{4.29}$$

By (4.26) and (4.28), as well as the condition (4.27), it can be verified that

$$Q(-\infty) < 0, \quad Q(0) > 0, \quad Q(\bar{H}_2) < 0, \quad Q(1) > 0, \tag{4.30a, b, c, d}$$

which show that the three roots $\bar{H}$ of the polynomial expression (4.29) are real and that there is only one root $\bar{H} \in (\bar{H}_2, 1)$. Since $\bar{H} \geqq H_2$ by (4.18), it follows from (4.29) and (4.30) that

$$\bar{H}_2 \leqq \bar{H} < 1, \quad H_2 \leqq H < H_1. \tag{4.31a, b}$$

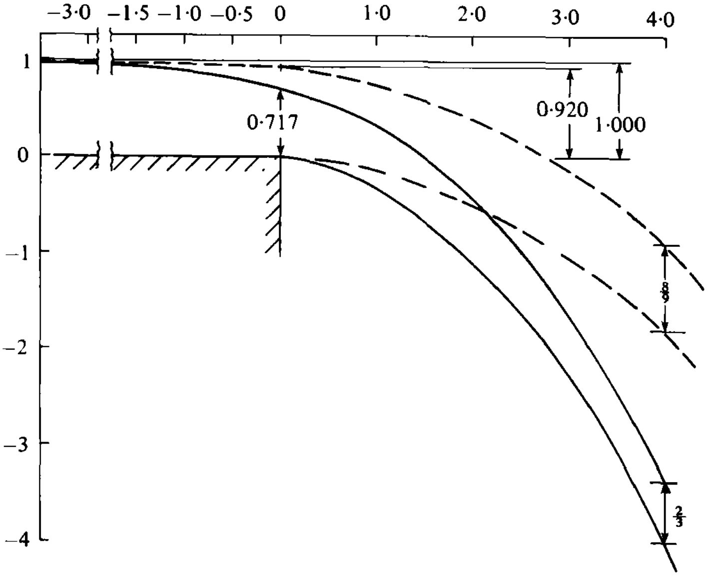

FIGURE 2. A plot of the solution exhibiting profiles of the fluid sheet for an upstream height $H_1 = 1$ meter and for two values of the Froude number F. Also indicated are the values in SI units of the height H at the edge of the fluid bed and the vertical thickness H_2 far downstream calculated from (4.36) and (4.26), respectively. ——, $F=1$; – – –, $F=2$.

We now integrate (4.10) to obtain the height of the free surface in the upstream region. Thus

$$\phi = H_1 - \frac{4H_1^3}{3(x+a)^2} \quad \text{for} \quad F = 1, \tag{4.32a}$$

$$\phi = H_1 - H_1(F^2-1)\,\mathrm{cosech}^2\left\{\left[\frac{3(F^2-1)}{4F^2}\right]^{\frac{1}{2}}\left(\frac{x+b}{H_1}\right)\right\} \quad \text{for} \quad F > 1, \tag{4.32b}$$

where the constants a and b are determined from the condition (4.11 a) and are given by

$$a = -H_1\left[\frac{4}{3(1-\bar{H})}\right]^{\frac{1}{2}}, \quad b = -H_1\left[\frac{4F^2}{3(F^2-1)}\right]^{\frac{1}{2}} \sinh^{-1}\left[\left(\frac{F^2-1}{1-\bar{H}}\right)^{\frac{1}{2}}\right]. \tag{4.33a,b}$$

This completes our development of the overfall problem. For convenience, we summarize the main results of the solution before a discussion of some of its features. In region I ($x \leqq 0$), the solution is

$$\phi u = k, \quad P = \rho^*\left[\tfrac{1}{2}gH_1^2 + k^2\left(\frac{1}{H_1} - \frac{1}{\phi}\right)\right], \quad \phi \text{ given by (4.32)}, \tag{4.34ab}$$

while in region II ($x > 0$) the solution is

$$\phi u = k, \quad P = \rho^* k^2\left(\frac{1}{H_2} - \frac{1}{\phi}\right), \quad \phi \text{ and } \psi \text{ given by (4.5) and (4.6)} \tag{4.35a,b}$$

and $\bar{H}$($\bar{H}_2 \leqq \bar{H} < 1$) is determined by (4.29) or equivalently by

$$F^2\bar{H}^3 + (3F^4 + 2F^2 + 1)\bar{H}^2 - 3(F^2 + 2F^4)\bar{H} + 3F^4 = 0. \tag{4.36}$$

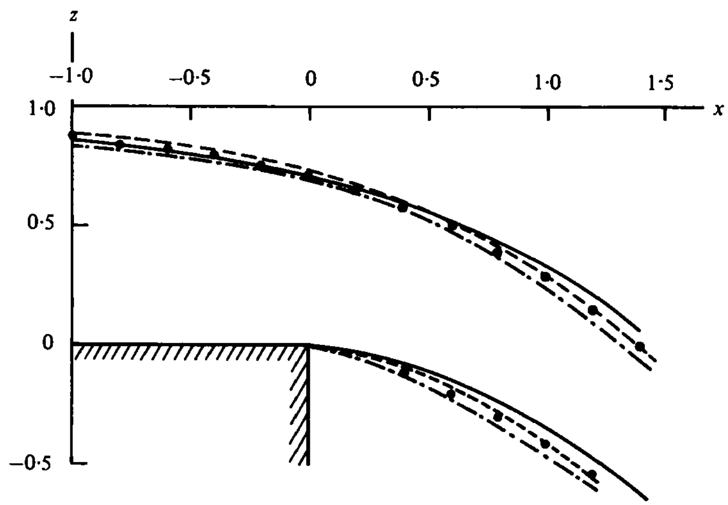

FIGURE 3. A comparison between the solutions and experimental results for the critical flow ($F = 1$) and upstream height $H_1 = 1$ m: ——, present solution; – – –, Chow & Han (1979), 40×40 grid; –·–·–, Southwell & Vaisey (1946), relaxation; ●, experimental results of Rouse (1936). (The results of other authors presented here are taken from figure 9 of Chow & Han 1979).

In the solutions for ϕ and ψ specified in (4.34)–(4.35), the constants a and b are given by (4.33), H_2 is calculated from (4.13), B is given by (4.5*b*) and the constants A, C, D are given by (4.16) and K in (4.11*b*) may be determined by differentiating the appropriate expression in (4.32) and evaluating ϕ_x at $x = 0$.

5. Discussion

It is of interest to solve (4.36) for F^2 in terms of $\bar{H}$. After rearranging (4.36), we have

$$3(1-\bar{H})^2F^4 - \bar{H}(1-\bar{H})(\bar{H}+3)F^2 + \bar{H}^2 = 0. \tag{5.1}$$

Although in general there are two roots of (5.1), only one of the roots corresponds to the overfall problem. The relevant root of (5.1) may be identified by recalling that when $F^2 = 1$, the only solution of (4.36) satisfying the restriction (4.31*a*) is $\bar{H} = 0{\cdot}71688$ (rounded off to 0·717). Since the solution of (5.1) must also predict that $F^2 = 1$ when $\bar{H} = 0{\cdot}717$, we may write

$$F^2 = \bar{H}\left\{\frac{\bar{H}+3-[(\bar{H}+3)^2-12]^{\frac{1}{2}}}{6(1-\bar{H})}\right\}. \tag{5.2}$$

Since $\bar{H} \geqq \bar{H}_2$ by (4.31*a*) and $\bar{H}_2$ has a minimum value $\frac{2}{3}$ (corresponding to $F = 1$ in (4.21), it follows that $\bar{H} \geqq \frac{2}{3}$ always and hence F^2 in (5.2) is always real. The simple formula (5.2) should be of practical interest in determining the flow rate from measurements of only the upstream height H_1 and the height H at the edge of the fluid bed.

By way of illustration, the solution obtained in §4 is depicted in figure 2 for two values of the Froude number $F = 1$ and $F = 2$ and for an upstream height $H_1 = 1$ m in SI units. Figure 3 compares the present solution for the critical flow ($F = 1$) with the corresponding numerical solutions of Chow & Han (1979) and Southwell & Vaisey

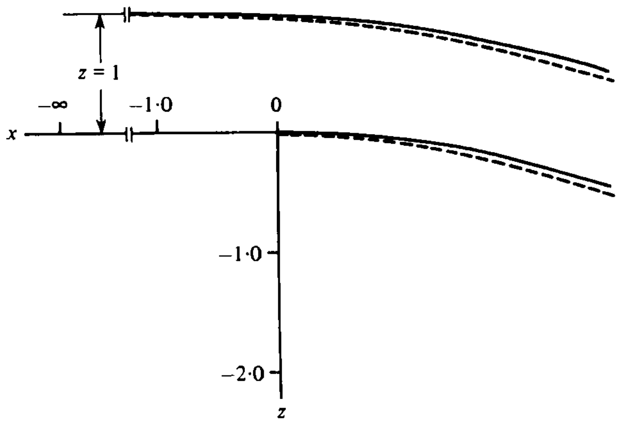

FIGURE 4. A comparison of the present solution (——) with the numerical solution of Chow & Han (1979) (- - -) for Froude number $F = 3$ and upstream height $H_1 = 1$ m.

(1946), as well as the experimental results of Rouse (1936). Inspection of figure 3 indicates that in the upstream region (region I in figure 1), the present solution is in good agreement with the experimental results. In particular, the present solution predicts a value $H = 0{\cdot}717$ m which is quite close to the experimental mean value† $H = 0{\cdot}716$ m reported by Rouse (1936, p. 260). Also in the full range of Froude numbers, the predictions of the present solution for the height H at the edge of the fluid bed are almost identical (to within $\frac{1}{2}$ %) to those obtained numerically from the three-dimensional equations by Markland and plotted in figure 8 of the Discussion of his paper (Markland 1965, p. 292). In addition, a comparison of the present solution with the numerical solution of Chow & Han (1979) for $F = 3$ is shown in figure 4. Inspection of figures 3 and 4 suggests that the differences between the present solution and that of Chow & Han (1979) decrease as the Froude number increases.

Returning to the solution presented in § 4 we note that although the jump conditions require the continuity of various quantities at the edge of the bed $x = 0$ (see equations (4.11) and (4.12)), certain discontinuities remain. In particular, it can be shown that the bottom pressure difference $\overline{P}$ is greater than zero at $x = 0^-$ and equal to zero at $x = 0^+$. It is also of interest to note that figure 10(c) of Chow & Han suggests that for $F = \sqrt{20}$ and $\bar{x} = 4{\cdot}45$ the horizontal velocity is fairly independent of the vertical coordinate. This result is consistent with the kinematical assumption (2.9), which requires the horizontal velocity to be independent of the vertical coordinate. As a further comparison, we mention that the formula for $\overline{H}_2$ in (4.26) is identical with one that can be derived from the exact three-dimensional equations of an incompressible, inviscid fluid under steady state conditions, as well as the assumptions of uniform flow upstream and uniform atmospheric pressure and constant horizontal velocity far downstream (see (6.29), p. 193 in Henderson 1966). Finally we note that the expression (4.6) may easily be used to show that the limit $x \to +\infty$ the fluid sheet becomes vertical as is to be expected. Of course, this limit may never be

† Rouse (1936, p. 260) reports that for the critical flow ($F = 1$) in the free overfall, he had found a 'mean value of 0·716, with an average arithmetic departure from 0·715 of 0·5 %'. For convenience we have converted the nondimensional value of $\overline{H}$ presented by Rouse (1936) to a dimensional value of $\overline{H}$ by multiplying by $H_1 = 1$ m.

attained experimentally since the fluid sheet must necessarily contact some solid boundary at some finite value of x.

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